Midterm exam Linear Algebra II

Thursday 07/03/2024, 18:30–20:30

1 (9 = 1 + 2 + 2 + 4 pts) **subspaces**

Let $\mathcal{V}$ be a vector space over $\mathbb{F}$. Consider two subspaces $\mathcal{U}$ and $\mathcal{W}$ of $\mathcal{V}$.

(a) Give an example to show that $\mathcal{U} \cup \mathcal{W}$ is in general not a subspace of $\mathcal{V}$.

(b) Prove that $\mathcal{U} \cup \mathcal{W}$ is a subspace of $\mathcal{V}$ if either $\mathcal{U} \subseteq \mathcal{W}$ or $\mathcal{W} \subseteq \mathcal{U}$.

(c) Now assume that $\mathcal{U} \cup \mathcal{W}$ is a subspace of $\mathcal{V}$. Suppose that there exists a vector $u \in \mathcal{U}$ such that $u \notin \mathcal{W}$. Show that for any vector $w \in \mathcal{W}$, either $u + w \in \mathcal{U}$ or $u + w \in \mathcal{W}$.

(d) Prove that if $\mathcal{U} \cup \mathcal{W}$ is a subspace of $\mathcal{V}$ then either $\mathcal{U} \subseteq \mathcal{W}$ or $\mathcal{W} \subseteq \mathcal{U}$.

2 (9 = 3 + 3 + 3 pts) **linear operators**

Consider the vector space P_2 of real polynomials of degree ≤ 2.

(a) Define the function $T : P_2 \to P_2$ by

$$(T(p))(x) := x^2 p(\tfrac{1}{x}) + p''(x).$$

Prove that T is a linear operator.

(b) Find a matrix representation of T with respect to the basis $(1, 1 + x, 1 - 2x + x^2)$.

(c) Are the following maps linear operators on P_2? Motivate your answers.

- $(R(p))(x) := xp(\tfrac{1}{x}) + p''(x)$.
- $(S(p))(x) := p(p''(x))$.

3 $(9 = 1 + 3 + 2 + 3$ pts) **change of basis**

Consider the vector space V over $\mathbb{C}$ consisting of all functions $f\colon \mathbb{Z}_{\geq 1} \to \mathbb{C}$. Here the sum of $f, g \in V$ is the function $f + g$ defined by $(f+g)(n) = f(n) + g(n)$, for all $n \in \mathbb{Z}_{\geq 1}$ and scalar multiplication (for $f \in V$ and $\lambda \in \mathbb{C}$) defined as $(\lambda f)(n) = \lambda f(n)$.

This exercise deals with the $\mathbb{C}$-subspace $W \subset V$ given by

$$W := \{f \in V \ : \ f(n+3) = f(n+2) - f(n+1) + f(n) \text{ for all } n \in \mathbb{Z}_{\geq 1}\}.$$

(a) Show that the function $h\colon \mathbb{Z}_{\geq 1} \to \mathbb{C}$ given by $h(n) = 1$ for any $n \geq 1$, is an element of W.

(b) Explain why the three functions $n \mapsto 1$ and $n \mapsto i^n$ and $n \mapsto (-i)^n$ form a basis β for W over $\mathbb{C}$.

(c) The linear map $S\colon W \to W$ is defined as follows: for any $f \in W$ we define $S(f)$ to be the following function: it sends any $n \in \mathbb{Z}_{\geq 1}$ to $f(n+1)$. Compute ${}_\beta[S]_\beta$, with β as in (b).

(d) Take $g \in W$ defined as $g(n) = \begin{cases} 1 & \text{if } 4 \text{ divides } n \text{ and if 4 divides } n+1, \\ 0 & \text{otherwise.} \end{cases}$
Then another basis for W is $\gamma = \{g, S(g), S(S(g))\}$. What is ${}_\beta[id_W]_\gamma$?
Here S is the map from part (c) and β is the basis given in (b).

4 $(9 = 2 + 2 + 2 + 3$ pts) **dimension, norms and inner products**

In this exercise, V is an inner product space over $\mathbb{R}$, with inner product of vectors $v, w \in V$ written as $\langle v, w\rangle$. We assume $0 \neq \dim V = n < \infty$ and we take a basis $\{v_1, \ldots v_n\}$ for V over $\mathbb{R}$. Let $T\colon V \to \mathbb{R}^n$ be the map defined by $T(v) = \begin{pmatrix} \langle v, v_1\rangle \\ \vdots \\ \langle v, v_n\rangle \end{pmatrix}$.

(a) Explain why T is linear over $\mathbb{R}$.

(b) Show that if $v \in \mathrm{Ker}(T)$, then $v \perp w$ for any w in the span of $\{v_1, \ldots, v_n\}$.

(c) Show that T is injective.

(d) Explain why the $n \times n$ matrix $(a_{i,j})$ with $a_{i,j} = \langle v_i, v_j\rangle$ is invertible.

4 pts free